

**The simplest form of the  $\kappa$  and  $\kappa^2$  order  
graviton self-interaction Lagrangian density  
in the weak field approximation**

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**Abstract**

We derive the simplest form of the  $\kappa(\kappa^2)$  order graviton self-interaction Lagrangian density  $\mathcal{L}_g^1(\mathcal{L}_g^2)$  in the weak field approximation. With the divergenceless condition, de Donder gauge and some combinatoric techniques, we derive the mathematically independent basis expressions for  $\mathcal{L}_g^1(\mathcal{L}_g^2)$  composed of 8(24) terms. By using the basis expressions  $\mathcal{L}_g^1(\mathcal{L}_g^2)$  is reduced to 6(18) terms.

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## 1. Introduction

Whenever one studies quantum gravity processes in the weak field approximation procedure [1], where the metric perturbation  $h_{\mu\nu}$  in  $g_{\mu\nu}$  from the flat space metric  $\eta_{\mu\nu}$  is identified with the gravitational field, one always encounters the difficulties in dealing with the complexity of the algebraic work which stems from the complicated nature of the graviton. A large number of indices appear in the interaction Lagrangian since a graviton has a pair of indices and it can even experience self-interaction. Furthermore, one should carry out numerous permutation and symmetrization operations on each index-pair and over all momentum-index triplets of the gravitons in the interaction Lagrangian to get the correct vertices. This is due to the symmetry property of the index pair of  $h_{\mu\nu}$  and the bosonic property of the graviton, respectively. In 1967, DeWitt [3] derived the graviton self-interaction Lagrangian up to 3- and 4- graviton vertices. This Lagrangian has been believed to be composed of the smallest terms but not been proved. Using this Lagrangian, cross-sections for the processes involving 3- or 4- graviton vertices have been evaluated [4,5] without further simplifying the Lagrangian except for the marginal correction of the sign of a term [4] in the 4- graviton vertex function. As shown in section 2, 3- and 4- graviton vertices can be obtained from  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$ .

In this paper, we present the basis expressions for  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  and the simplest form of  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  in terms of the basis expressions.  $\mathcal{L}_g^1$  is simplified from 11 to 6 in the number of terms. For the case of  $\mathcal{L}_g^2$ , 28 terms are reduced into only 18 terms. Though the numbers of terms are still rather large, they cannot be simplified any more because they are expressed by the basis expressions. Using our new  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$ , we obtained the same differential cross-sections for the graviton-scalar( $gs \rightarrow gs$ ) [4], graviton-fermion( $gf \rightarrow gf$ ) [5], graviton-massive vector boson( $gW \rightarrow gW$ ) [4], graviton-photon( $g\gamma \rightarrow g\gamma$ ) [4] and graviton-graviton( $gg \rightarrow gg$ ) [4] elastic scatterings with those of the corresponding references. In this paper, we review the formal procedure to obtain the graviton self-interaction Lagrangian in the weak field approximation in section 2. To deal with the combinatoric properties of the typical terms

in the Lagrangian conveniently, we introduce our index convention which is also helpful in computer-aided calculations in section 3. Sections 4 and 5 are devoted to explain our strategy for getting the simplest expressions. In these sections, the scheme of constructing basis expressions for  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  and also the explicit forms of bases as well as the simplest forms of  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  are given in these sections. Conclusion is made in section 6.

## 2. Analysis of the problem

In the classical theory of general relativity [2], the action for pure gravity is given by

$$I_g = 2\kappa^{-2} \int d^4x \sqrt{-g(x)} R(x), \quad (1)$$

where  $\kappa = \sqrt{32\pi G_N}$  with the Newtonian constant  $G_N$ ,  $R(x)$  the scalar curvature, and  $g(x) = \det g_{\mu\nu}$ , i.e., the graviton self-interaction Lagrangian density is given by

$$\mathcal{L}_g = 2\kappa^{-2} \sqrt{-g} R. \quad (2)$$

In the weak field approximation, one can write the metric tensor  $g_{\mu\nu}$  as the sum of the flat space metric  $\eta_{\mu\nu}$  and a metric perturbation  $h_{\mu\nu}$  as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \quad (3)$$

Here  $h_{\mu\nu}$  can be identified with the gravitational field by giving  $[\text{mass}]^{-1}$  dimension to the parameter  $\kappa$ . The curvature scalar  $R$  is defined by

$$\begin{aligned} R &= g^{\mu\nu} R_{\mu\nu} \\ &= g^{\mu\nu} R_{\mu\lambda\nu}^{\lambda}, \end{aligned} \quad (4)$$

where  $R_{\mu\nu}$  is the Ricci tensor and the Riemann-Christoffel curvature tensor  $R_{\mu\kappa\nu}^{\lambda}$  is defined by

$$R_{\mu\kappa\nu}^{\lambda} = \partial_{\nu} \Gamma_{\mu\kappa}^{\lambda} - \partial_{\kappa} \Gamma_{\mu\nu}^{\lambda} + \Gamma_{\mu\kappa}^{\alpha} \Gamma_{\alpha\nu}^{\lambda} - \Gamma_{\mu\nu}^{\alpha} \Gamma_{\alpha\kappa}^{\lambda}, \quad (5)$$

and the affine connection  $\Gamma_{\mu\nu}^{\lambda}$  is defined by

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\alpha}(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}) \quad (6)$$

By using the identity  $g^{\mu\lambda}g_{\lambda\nu} = \delta_\nu^\mu$ , we get the  $\kappa$  expansion of  $g^{\mu\nu}$ ,

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h_\alpha^\nu - \kappa^3 h^{\mu\alpha} h_\alpha^\beta h_\beta^\nu + \dots \quad (7)$$

Expanding  $g$  and  $\sqrt{-g}$  up to  $\kappa^3$  order, we get

$$g = -1 - \kappa h - \frac{1}{2}\kappa^2(h^2 - h^{\mu\nu}h_{\mu\nu}) - \frac{1}{6}\kappa^3(h^3 - 3hh^{\nu\sigma}h_{\nu\sigma} + 2h^{\mu\nu}h_\nu^\sigma h_{\sigma\mu}), \quad (8)$$

$$\sqrt{-g} = 1 + \frac{1}{2}\kappa h + \frac{1}{8}\kappa^2(h^2 - 2h^{\mu\nu}h_{\mu\nu}) + \frac{1}{48}\kappa^3(h^3 - 6hh^{\nu\sigma}h_{\nu\sigma} + 8h^{\mu\nu}h_\nu^\sigma h_{\sigma\mu}), \quad (9)$$

with the definition  $h = h_\mu^\mu$ . Then we can obtain the leading term of  $\mathcal{L}_g$  as

$$\mathcal{L}_g = 2\kappa^{-2}[\kappa^2(-\frac{1}{8}\partial_\mu h \partial^\mu h + \frac{1}{4}\partial_\mu h_\beta^\alpha \partial^\mu h_\alpha^\beta) + O(\kappa^3)], \quad (10)$$

where we use the harmonic(de Donder) gauge,

$$\partial_\mu h_\alpha^\mu = \frac{1}{2}\partial_\alpha h, \quad (11)$$

and omit total derivatives. Then  $\kappa$  expansion of Lagrangian density is given by

$$\begin{aligned} \mathcal{L}_g &= \frac{2}{\kappa^2}\sqrt{-g}R \\ &= \mathcal{L}_g^0 + \kappa\mathcal{L}_g^1 + \kappa^2\mathcal{L}_g^2 + \dots \end{aligned} \quad (12)$$

The leading term  $\mathcal{L}_g^0$  is obviously of the simplest form as

$$\mathcal{L}_g^0 = -\frac{1}{4}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu h_\beta^\alpha \partial^\mu h_\alpha^\beta. \quad (13)$$

$\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  can be arranged into simpler forms by using some conditions explained below. According to Refs. [3,4], they are reduced into 11 and 28 terms, respectively. But we have obtained still more simplified forms of  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  and proved that they cannot be reduced any more.

The framework of our proof is as follows: Let  $\mathcal{E}^i (i = 1, 2)$  be the space composed of any mathematically possible expressions which has the typical form of the terms inside  $\mathcal{L}_g^i$ . (in

this paper any upper indices  $i, j, k, l$  are related only to the order of  $\kappa$ .) To get the simplest form of  $\mathcal{L}_g^i$ , we should find out the basis  $B^i$  for  $\mathcal{E}^i$ . We start our search for  $B^i$  from a set  $C^i$  which contains all the possible terms inside any expression in  $\mathcal{E}^i$ . Of course  $C^i$  spans  $\mathcal{E}^i$  and has all the possible permutaions of indices for the typical terms. Though  $C^i$  has a large number of elements, it is both countable and finite. Without any condition and neglecting the multiplication commutativity, the numbers of elements of  $C^i$ 's are

$$\begin{aligned} C^1 &: \frac{8!}{(2!)^4 \cdot 4!}, \\ C^2 &: \frac{10!}{(2!)^5 \cdot 5!}. \end{aligned} \tag{14}$$

With the symmetry property of  $h_{\mu\nu}$  and de Donder gauge condition, we can reject a lot of duplicate elements from  $C^i$ . If  $C^i$  has  $n^i$  elements in this step, we can express  $C^i$  as

$$C^i = \{c_1^i, c_2^i, c_3^i, \dots, c_{n^i}^i\}. \tag{15}$$

We can obtain the useful relations from the fact that the total derivative of the Lagrangian can be neglected. Since any  $c_j^i$  has derivative factors, we can rewrite it by partial integration neglecting the total derivatives. In general the result of this partial integration is also an element of  $\mathcal{E}^i$ . Therefore, we can express each  $c_j^i$  as an linear combination of  $c_k^i$ 's ( $k = 1, 2, 3, \dots, n^i$ ) as

$$c_j^i = \sum_{k=1}^{n^i} d_{jk}^i c_k^i, \tag{16}$$

where  $d_{jk}^i$ 's are pure numbers. If we solve these  $n^i$ -simultaneous equations, we can find the basis. Not all the equations are independent and some  $c_j^i$ 's are rewritten by itself under partial integration as

$$c_l^i = c_l^i. \tag{17}$$

If there are  $m^i$  non-trivial equations in the  $n^i$ -simultaneous equations, there exist  $n^i - m^i$  free variables in the simultaneous equations. Therefore we can take any  $n^i - m^i$  elements from  $C^i$  to make a basis set  $B^i$ . It is a common orthogonalization procedure. Getting the

basis expressions, we can obtain the simplest form of the Lagrangians by expressing the original  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  in terms of these basis expressions. Let us find out them step by step in the following sections.

### 3. Convention Summary

First of all, we introduce our new convention for expressions appearing in  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$ . Through partial integration and omitting total derivative, every term of  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  can be transformed into the following typical forms,

$$\begin{aligned}\mathcal{L}_g^1 &: \partial_{\gamma} h_{\gamma\gamma} \partial_{\gamma} h_{\gamma\gamma} h_{\gamma\gamma} \\ \mathcal{L}_g^2 &: \partial_{\gamma} h_{\gamma\gamma} \partial_{\gamma} h_{\gamma\gamma} h_{\gamma\gamma} h_{\gamma\gamma},\end{aligned}\tag{18}$$

where  $\gamma$ 's are the unknown indices. There are 4 and 6 pairs of indices in the typical terms of  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$ , respectively. Here we neglect the difference between the upper and lower indices because every pair of indices is composed of 1 upper index and 1 lower one - contraction is always carried out between them. To show the combinatoric characteristics of the indices, we use a number in place of a Greek character as an index. Here is an explicit example of our convention.

$$\begin{aligned}h_{\mu\nu} &\rightarrow (12) \\ \partial_{\alpha} h_{\mu\nu} &\rightarrow (3(12))\end{aligned}\tag{19}$$

Distinct numbers should be given to the different indices. Innermost parenthesis means  $h_{\gamma\gamma}$ . If there is a double parenthesis, outer one means derivative. Every number in the parenthesis means the correspondent index. For brevity, double derivative expression like  $(1(2(34)))$  is written as  $(12(34))$ . This convention is very convenient to read and write in the computer-aided calculations. Especially, it shows the combinatoric characteristics of each expressions much more clearly.

#### 4. $\mathcal{L}_g^1$ reduction

For the case of  $\mathcal{L}_g^1$ , the general form is  $(a(l_1 l_2))(b(m_1 m_2))(n_1 n_2)$ . After listing all possible combinations of indices made of 4 pairs of numbers, we can find the projection among the expressions by using partial integration and neglecting total derivatives as

$$\begin{aligned}
& (a(l_1 l_2))(b(m_1 m_2))(n_1 n_2) \\
&= - (l_1 l_2)(ab(m_1 m_2))(n_1 n_2) - (l_1 l_2)(b(m_1 m_2))(a(n_1 n_2)) \\
&= + (b(l_1 l_2))(a(m_1 m_2))(n_1 n_2) - (l_1 l_2)(b(m_1 m_2))(a(n_1 n_2)) \\
&+ (l_1 l_2)(a(m_1 m_2))(b(n_1 n_2)).
\end{aligned} \tag{20}$$

Throughout this procedure we extensively use the de Donder gauge condition to get the factor such as (11) or (1(22)) maximally. Rejecting the dependent expressions one by one, we get the basis expressions which can only be transformed into itself. They are given by

$$\begin{aligned}
f_1^1 &= (1(22))(1(33))(44) = \partial^\mu h^\nu{}_\nu \partial_\mu h^\alpha{}_\alpha h^\beta{}_\beta \\
f_2^1 &= (1(22))(4(33))(14) = \partial^\mu h^\nu{}_\nu \partial^\beta h^\alpha{}_\alpha h_{\mu\beta} \\
f_3^1 &= (1(22))(1(34))(34) = \partial^\mu h^\nu{}_\nu \partial_\mu h^{\alpha\beta} h_{\alpha\beta} \\
f_4^1 &= (1(22))(3(14))(34) = \partial^\mu h^\nu{}_\nu \partial^\alpha h_\mu{}^\beta h_{\alpha\beta} \\
f_5^1 &= (1(23))(1(23))(44) = \partial^\mu h^{\nu\alpha} \partial_\mu h_{\nu\alpha} h^\beta{}_\beta \\
f_6^1 &= (1(23))(4(23))(14) = \partial^\mu h^{\nu\alpha} \partial^\beta h_{\nu\alpha} h_{\mu\beta} \\
f_7^1 &= (1(23))(2(34))(14) = \partial^\mu h^{\nu\alpha} \partial_\nu h_\alpha{}^\beta h_{\mu\beta} \\
f_8^1 &= (1(23))(1(24))(34) = \partial^\mu h^{\nu\alpha} \partial_\mu h_\nu{}^\beta h_{\alpha\beta}.
\end{aligned} \tag{21}$$

The basis is made up of 8 expressions. This number is smaller than that(11) of the old  $\mathcal{L}_g^1$  [4]. If one expresses  $\mathcal{L}_g^1$  in terms of this basis, it is composed of even smaller number of terms, i.e., 6 terms as

$$\mathcal{L}_g^1 = -\frac{1}{8}f_1^1 + \frac{1}{2}f_3^1 + \frac{1}{4}f_5^1 - \frac{1}{2}f_6^1 + f_7^1 - f_8^1. \tag{22}$$

## 5. $\mathcal{L}_g^2$ reduction

The method of  $\mathcal{L}_g^2$  reduction is basically the same as that of  $\mathcal{L}_g^1$ . The general form of a term in  $\mathcal{L}_g^2$  is of the form,  $(a(l_1 l_2))(b(m_1 m_2))(n_1 n_2)(r_1 r_2)$ , which has one more pair of indices with one more  $h_{??}$ . There are the same number of derivative factors which make the transformation rule alike. We can find the projection of a typical term to the others by using partial integration and neglecting total derivatives as

$$\begin{aligned}
& (a(l_1 l_2))(b(m_1 m_2))(n_1 n_2)(r_1 r_2) \\
&= -(l_1 l_2)(ab(m_1 m_2))(n_1 n_2)(r_1 r_2) \\
&\quad -(l_1 l_2)(b(m_1 m_2))(a(n_1 n_2))(r_1 r_2) \\
&\quad -(l_1 l_2)(b(m_1 m_2))(n_1 n_2)(a(r_1 r_2)) \\
&= +(b(l_1 l_2))(a(m_1 m_2))(n_1 n_2)(r_1 r_2) \\
&\quad +(l_1 l_2)(a(m_1 m_2))(b(n_1 n_2))(r_1 r_2) \\
&\quad +(l_1 l_2)(a(m_1 m_2))(n_1 n_2)(b(r_1 r_2)) \\
&\quad -(l_1 l_2)(b(m_1 m_2))(a(n_1 n_2))(r_1 r_2) \\
&\quad -(l_1 l_2)(b(m_1 m_2))(n_1 n_2)(a(r_1 r_2)). \tag{23}
\end{aligned}$$

In this case, we can find the projections of a typical term to 5 other terms. Rejecting the dependent expressions, we get the basis expressions,

$$\begin{aligned}
f_1^2 &= (1(22))(1(33))(44)(55) = \partial^\mu h^\nu{}_\nu \partial_\mu h^\alpha{}_\alpha h^\beta{}_\beta h^\sigma{}_\sigma \\
f_2^2 &= (1(22))(1(33))(45)(45) = \partial^\mu h^\nu{}_\nu \partial_\mu h^\alpha{}_\alpha h^{\beta\sigma} h_{\beta\sigma} \\
f_3^2 &= (1(22))(3(44))(13)(55) = \partial^\mu h^\nu{}_\nu \partial^\alpha h^\beta{}_\beta h_{\mu\alpha} h^\sigma{}_\sigma \\
f_4^2 &= (1(22))(3(44))(15)(35) = \partial^\mu h^\nu{}_\nu \partial^\alpha h^\beta{}_\beta h_\mu{}^\sigma h_{\alpha\sigma} \\
f_5^2 &= (1(22))(1(34))(34)(55) = \partial^\mu h^\nu{}_\nu \partial_\mu h^{\alpha\beta} h_{\alpha\beta} h^\sigma{}_\sigma \\
f_6^2 &= (1(22))(1(34))(35)(45) = \partial^\mu h^\nu{}_\nu \partial_\mu h^{\alpha\beta} h_\alpha{}^\sigma h_{\beta\sigma} \\
f_7^2 &= (1(22))(3(14))(34)(55) = \partial^\mu h^\nu{}_\nu \partial^\alpha h_\mu{}^\beta h_{\alpha\beta} h^\sigma{}_\sigma
\end{aligned}$$



$$\begin{aligned}
f_8^2 &= (1(22))(3(14))(35)(45) = \partial^\mu h^\nu{}_\nu \partial^\alpha h_\mu{}^\beta h_\alpha{}^\sigma h_{\beta\sigma} \\
f_9^2 &= (1(22))(3(45))(45)(13) = \partial^\mu h^\nu{}_\nu \partial^\alpha h^{\beta\sigma} h_{\beta\sigma} h_{\mu\alpha} \\
f_{10}^2 &= (1(22))(3(45))(41)(53) = \partial^\mu h^\nu{}_\nu \partial^\alpha h^{\beta\sigma} h_{\beta\mu} h_{\sigma\alpha} \\
f_{11}^2 &= (1(23))(2(13))(45)(45) = \partial^\mu h^{\nu\alpha} \partial_\nu h_{\mu\alpha} h^{\beta\sigma} h_{\beta\sigma} \\
f_{12}^2 &= (1(23))(1(23))(44)(55) = \partial^\mu h^{\nu\alpha} \partial_\mu h_{\nu\alpha} h^\beta{}_\beta h^\sigma{}_\sigma \\
f_{13}^2 &= (1(23))(1(23))(45)(45) = \partial^\mu h^{\nu\alpha} \partial_\mu h_{\nu\alpha} h^{\beta\sigma} h_{\beta\sigma} \\
f_{14}^2 &= (1(23))(1(24))(34)(55) = \partial^\mu h^{\nu\alpha} \partial_\mu h_\nu{}^\beta h_{\alpha\beta} h^\sigma{}_\sigma \\
f_{15}^2 &= (1(23))(1(24))(35)(45) = \partial^\mu h^{\nu\alpha} \partial_\mu h_\nu{}^\beta h_\alpha{}^\sigma h_{\beta\sigma} \\
f_{16}^2 &= (1(23))(1(45))(23)(45) = \partial^\mu h^{\nu\alpha} \partial_\mu h^{\beta\sigma} h_{\nu\alpha} h_{\beta\sigma} \\
f_{17}^2 &= (1(23))(1(45))(24)(35) = \partial^\mu h^{\nu\alpha} \partial_\mu h^{\beta\sigma} h_{\nu\beta} h_{\alpha\sigma} \\
f_{18}^2 &= (1(23))(4(23))(14)(55) = \partial^\mu h^{\nu\alpha} \partial^\beta h_{\nu\alpha} h_{\mu\beta} h^\sigma{}_\sigma \\
f_{19}^2 &= (1(23))(4(23))(15)(45) = \partial^\mu h^{\nu\alpha} \partial^\beta h_{\nu\alpha} h_\mu{}^\sigma h_{\beta\sigma} \\
f_{20}^2 &= (1(23))(4(25))(13)(45) = \partial^\mu h^{\nu\alpha} \partial^\beta h_\nu{}^\sigma h_{\mu\alpha} h_{\beta\sigma} \\
f_{21}^2 &= (1(23))(4(25))(14)(35) = \partial^\mu h^{\nu\alpha} \partial^\beta h_\nu{}^\sigma h_{\mu\beta} h_{\alpha\sigma} \\
f_{22}^2 &= (1(23))(2(34))(15)(45) = \partial^\mu h^{\nu\alpha} \partial_\nu h_\alpha{}^\beta h_\mu{}^\sigma h_{\beta\sigma} \\
f_{23}^2 &= (1(23))(2(34))(14)(55) = \partial^\mu h^{\nu\alpha} \partial_\nu h_\alpha{}^\beta h_{\mu\beta} h^\sigma{}_\sigma \\
f_{24}^2 &= (1(23))(2(45))(14)(35) = \partial^\mu h^{\nu\alpha} \partial_\nu h^{\beta\sigma} h_{\mu\beta} h_{\alpha\sigma}. \tag{24}
\end{aligned}$$

The basis is made up of 24 expressions. This number is smaller than that(28) of the old  $\mathcal{L}_g^2$ . If one expresses  $\mathcal{L}_g^2$  in terms of this basis, it is composed of even smaller number of terms , i.e., 18 terms as

$$\begin{aligned}
\mathcal{L}_g^2 &= -\frac{1}{32}f_1^2 + \frac{1}{8}f_2^2 + \frac{1}{4}f_5^2 - \frac{1}{2}f_6^2 + \frac{1}{2}f_8^2 - \frac{1}{4}f_{11}^2 + \frac{1}{16}f_{12}^2 - \frac{1}{8}f_{13}^2 - \frac{1}{2}f_{14}^2 \\
&\quad + f_{15}^2 - \frac{1}{2}f_{16}^2 + \frac{1}{2}f_{17}^2 - \frac{1}{4}f_{18}^2 + \frac{1}{2}f_{19}^2 - f_{20}^2 + f_{21}^2 - 2f_{22}^2 + \frac{1}{2}f_{23}^2. \tag{25}
\end{aligned}$$

This reduction in the number of terms in  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  can not be done any further. The simplest form shows her face after 27 year's latency. Though the idea we employed to get

the results is somewhat simple, the actual procedure are hard to be carried out properly if one sticks to manual work only. Throughout our calculation, we extensively used a symbolic manipulation package *Mathematica* [6] which enabled us to execute the powerful pattern matching programs.

## 6. Conclusion

The simplified forms of the graviton self-interaction Lagrangian densities are derived to the  $\kappa$  and  $\kappa^2$  order and they are proved to be the simplest. The key point is that we first construct the basis expressions for the  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  and express them by this basis. We presented these basis sets explicitly and described not only the practical way to get the basis but also the formal proof. We also checked our results by calculating the observables of several real processes. To check the 3-graviton vertex, we evaluated the differential cross-sections for the processes, graviton-scalar( $gs \rightarrow gs$ ) [4], graviton-fermion( $gf \rightarrow gf$ ) [5], graviton-massive vector boson( $gW \rightarrow gW$ ) [4], graviton-photon( $g\gamma \rightarrow g\gamma$ ) [4], by using our simplest  $\mathcal{L}_g^1$ . The differential cross-section of the process graviton-graviton( $gg \rightarrow gg$ ) [4] is also evaluated in the same way to check the 3-and 4-graviton vertices. We obtained the same results as those of the previous graviton self-interaction Lagrangian. With the simplified graviton self-interaction Lagrangian density, we can save much time and avoid errors in dealing with the numerous indices. And our new method to find the basis expressions for the  $\mathcal{L}_g^1$  and  $\mathcal{L}_g^2$  can be applied to any order of  $\kappa$  with slight modification. We presented a successful application of the powerful pattern matching abilities of the symbolic manipulation package *Mathematica* to dealing with the physical problems involving complicated index relations.

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